On Voronoi’s Method of Reducing Quadratic Forms

By Dr. Andrew W. Harrell

Georg Voronoi was a Russian number theorist who living in St. Petersburg at the turn of the last century. He studied in the field called the “geometry of numbers” under the famous Polish mathematician Hermann Minkowski (who is known to physicists as the inventor of Minkowski space-time). They were interested in such topics as how to use continued fractions and algebraic numbers in order to determine “best approximations” to irrational and trancendental numbers. The field of algebraic number theory had just been invented at that time by famous theoretical mathematicians like Dedekind, Kronecker, and Hilbert. Also, the new areas of mathematical study called group theory which studied the symmetries of space were related to his research

- Given a positive definite quadratic form we diagonalize such a form by writing its matrix in the form trans(P)*P where P(x)= (chi_1,…chi_n). Then, the translates of the sphere defined by phi(x) = chi_1^2 +,…+chi_n^2 <= M/4, M = the minimum of phi for integral variable values, by the vectors of the lattice chi_1,…chi_n form a sphere packing. Minkowski in reference [11] proved that no form in dimension 3 not equivalent to that associated with the rhombic dodecahedron gives as large a value as pi/sqrt(18).

- In 1907 and 1908 G.F. Voronoi introduced two methods of reducing positive define quadratic forms, in a series of three articles in Crelle's journal [12]. The first method used the inner product on the corresponding space of n-dimensional symmetric matrices. The second method involved what he called primitive parallelohedrons (in which each vertex and face of the form's lattice belongs to exactly one edge and one face respectively of all translates and each face corresponds to exactly one face of all the other bodies). Two primitive parallelohedra are said to be of the same type if they determine the same lattice. If each primitive paralleohedron is associated to the set of points closer to the origin than any other lattice in the metric defined by the form. Then, generalizing Dirichlet's method of reduction, he subdivided the sapce of all positive definite forms into cones which correspond to a given type of primitive parallelohedra. One can then check all the forms corresponding to neighboring parallelohedra in the lattice in order to fill out a fundamental region in the whole space for the subgroup of reduced forms. John H. Conway explains this in more detail for the 3 dimensional lattices of forms in his books [1] and [3]. And, he and Neil Sloane completed the task of classification of all low dimensional lattices of this type in the series of papers published in 1988 until 1997 in the Journal of the Royal Society of London.

- In 1974 I wrote a thesis [10] while studying with Dr. Ichiro Satake at UC Berkeley which includes a short introduction and exposition of these ideas , a clarified presentation of the relation between Voronoi's two methods in the case n=3 and a short discussion of the application of these geometric ideas to the algebrais problem of desingularization of the compactified domain associated with Siegel's modular functions of degree 2 and 3.
- A few years after this some people studying computational geometry programs in computer science at MIT developed a short computer algorithm that uses Voronoi's method of reduction, and a given set of statistical data in n-space, to compute a subdivision of the sample space into partitions of influence of the nearest neighbors of the points. This has applications to plotting terrain surfaces, target recognition identification problems arising from discretely sampled measured data points. It also has applications to computing discrete element approximations in numerical partial differentiation programs. Nowadays, if you do an internet google search on Voronoi triangulation or "Voronoi diagram" there are tutorials and applets which you can download which explain this area of research in computational geometry much better than when I wrote my thesis on it in 1974. In fact, then the whole field of computer graphics and its underlying mathematics didn't really exist. What a difference 35 years can make in computer science and mathematics.
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- The basic idea of Voronoi was that to determine best sphere packings is essentially the same problem as to plot the regions of nearest neighbor points in a statistical sample of discrete data. Using, the metric of the discrete lattice of the sample space we draw out the fundamental domains of the regions of the packing. Later in computer science this was called the Voronoi region of nearest neighbor data point. If we are interested in dividing the space of positive definite metrics itself into fundamental regions we canstart out with a “self-dual” central form. We can then express the coordinates of all the positive definite forms in the total space in terms of the group of all lattice transformations of all the lattices’ forms. By reflecting the what Voronoi defined as the “normal coordinates” in the face of the extremal body gives the neighboring forms which are also in the fundamental domain of reduced forms found around a central, self-dual or “perfect” form. The mathematician H.M.S. Coxeter has written several fascinating books explained how the details of this is connected to determining group symmetries of higher dimensional polytopes. The recent book of Grunbaum[9] is also a good reference. It also explains some applications of these ideas to the field of convex geometry and integer programming.

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