How do We Define the number One?

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This question arises when we consider how we can develop a better understanding of the interrelations of science and faith. At the turn of the last century work on the area of the foundations of mathematical analysis and the beginnings of the development of mathematical logic increased. This happened along with the invention of digital computers. And, a new area of mathematical area of research called set theory was created in order to understand what "a real number" in Calculus means. Leopold Kronecker made his famous statement, "God created the integers and all else is the work of man." But, how did God create the integers? Plato's dialogue Parmenides is perhaps his hardest to understand work and the most important attempt in the classical era to try understand different ways we can answer this question. What is a set? What is an empty set (basically this is determined logically when you know what an element in a set is and what a set is)? This talk will give a short history of some of the progress mathematicians and logicians have made trying to answer these questions since the beginning of the last century. We have shown, that except for some notable gaps, how "real numbers (rational, algebraic, transcendental)", and likewise various other "complex and ideal numbers" can all be constructed logically from the positive integers. The possibility of the "notable gaps" come from the proof of the independence of the continuum hypothesis.

- SOME REASONS FOR THINKING ABOUT THIS QUESTION
- In terms of its relation to metaphysics and epistemology: The Greek philosophers realized this as a fundamental philosophical question too. Plato's dialogue Parmenides is perhaps his hardest and most important attempt in the classical era to try and understand this . And, it deals with just this question, 'What is does the Concept of One mean philosophically and mathematically?'
- "Among different languages, even where we cannot suspect the least connexion or communication, it is found, that the words, expressive of ideas, the most compounded, do yet nearly correspond to each other: a certain proof that the simple ideas, comprehended in the compound ones, were bound together by some universal principle, which had an equal influence on all mankind." A Treatise on Human Understanding Book 1 Section III
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- Isn't this Hume saying that he believes in such a thing as a "concept of a concept"?
  - How can we understand what the deeper philosophic question of What a "Concept of a Concept" is if we don't understand what the Concept of One is? This basic question is phrased above as a mathematical question in the logical foundations of the branch of mathematics called set theory. We believe the question of how we can develop a better understanding of science and faith is interconnected to this question. Thus, it is not only its mathematical implications that make this conundrum important for us to solve.
    This question from mathematical research relates to a fundamental theological and spiritual one… If we are to say that the fundamental nature of God is that He is One, what do we mean by this? The standard theological answer to this question is contained in a religion's teaching about the name of God (which is a trinity for Christians and Hindus). Is it possible to be a Father and Mother of God using structures of thinking within His Holy Spirit operating in our own minds and spirits and souls? Elsewhere on an internet prayer website I have posted some thoughts about this (see references). And, an understanding of how our human interest in this question as a mathematical one developed will certainly better help us understand how our different human forms of science and faith are related in us. To do this we need to go back to the turn of the last century when questions of the foundations of the area of mathematical analysis and the beginnings of the development of mathematical logic as it relates to the invention of computers was being developed.

A Short History of the Number One

- There is a documentary called "The story of One," made by Terry Jones (a Monty Python member). Quite interesting...
- " 20,000 years ago the number one exists for the first time.
- This is determined from evidence of human scratchs on bones. Many human societies, like the aboridigines, for example, never and still don't to this day, use any numbers (or even the number one).
- However, the whole science of measurement depends on having an idea of what the number one means to start out the measurements. We know that the Egyptians were some of the first to develop new methods for measurements (using a ruler) and hence beginning a question of what one means inside of us." [8]

- "Later in human history, the important Greek philosopher Pythagoras set up a group of vegetarian philosophers and mathematicians. He believed everything, especially including music, was made of numbers. He wanted to understand why certain combinations of notes sound harmonious. He studied ratios of whole number (collections of multiples of one) in order to understand this. He coined the term, "music of the spheres". If the beauty of music relies on whole numbers then so must everything else. And, since whole numbers are at the heart of music and one is at the heart of whole numbers it must be very important to understand what "one" is. However, the rationale for this belief system was later destroyed by the discovery of "irrational numbers". Pythagoras could not conceive of numbers unless they represent actual objects. Plato, and later Frege, believed that "numbers" were mental objects.
- Plato in the dialogue Parmenides starts humankind off studying "How to We (or You) Define the Number One." In this dialogue Socrates and Parmenides discuss the arguments and paradoxes of Zeno and other contemporary Greek Philosophers. It assumes a knowledge of previous dialogues like Phaedrus where Plato has explained his theory of independently existing mental objects called Forms. How do we "know" these independently existing mental objects? Our mind can know them by "participating" in them, not in the sense that through sense experience we collect sense-contents of material objects but in another more directly intuitive sense.

These questions are about something Plato believes in existing as a Form or Concept in Divine mind and calls "The One"

How does Plato use Concepts to Investigate Concepts in this dialogue:
.... We trace out the definition of a lessor known concept by recognizing in it the same elements as are present in a better known concept. In literature this method is used in the form of metaphors and similies. In ordinary speech it is the method of analogies.
....Since we are actually hunting concepts, we can bring one out in the open, so to speak, by asking questions about. These questions will expose what dols the concept together.
.... By seeking out those elements in concepts that constantly reoccur, we obtain the most economical description of them.

THE MIND AS A SWITCHING YARD----PASCAL "ONE MUST SUBSTITUTE DEFINITION FOR THE DEFINED."
Names are associated with objects; so are their meaning. But an objects's name can be one the tip of the tongue when a corresponding meaning isn't: how else do we know sounds when we hear them, or other things when we see
them. The answer is that our mind automatically deals with things by sorting them into pre-established groups. We have, in our mind, a switching yard. If we want to know what a song or anything else is, we must ask ourselves question about the mental defintion of it. It will be helpful to discuss the nature of this mental switching yard, in order to understand how to ask these questions.

A concept (and hence the Concept of One that we are trying to

- This dialogue discusses the question, "What is the Form of One" (if indeed such a thing exists… for the method used is to discuss a philosophic problem by both assuming the consequences of believing a logical proposition about Forms to be True and then also believing it to be false. This method of mental analysis anticipated that of Boolean logic functions by several thousand years. So, during the dialogue a series of eight "hypotheses" are put forward. And, the "participants" in the dialogue discuss the consequences of assuming the hypotheses are true or not. The first two hypotheses are that.

- However, after this, Archimedes modified this philosophic assumption somewhat by telling us we could think of numbers as objects (concepts) in themselves. This tended to take "one" away from being the "essence of the universe". "[8]

Notes on Plato's dialogue the Parmenides:

These notes can only be cursory here because of the deep and complicated nature of this wonderful treasure chest of a book.

A primary reference and starting point for further study is "Plato and Parmenides", by Francis MacDonald Cornford, Bobbs-Merrill.

Important note: Trying to read and understand the translation by itself without any explanation, eg as given in Great Books volume 7 U. of Chicago Press , is almost impossible in my opinion

Also, the One under discussion here is not necessarily the "Number One" of FR

From passage 137C

Hypothesis I) "If the One is defined as absolutely one, it is in no sense defined many or a whole of parts".

This hypothesis turns out to have the consequences that assuming this:

137D "the One has no parts and is unlimited"

138A "it has no extension or shape"

139B "is neither in motion or in rest"

139E-140D "it is not the same or different, like or unlike, equal or unequal from itself"

"it cannot be or become older and is in no sense an 'is'"

From passage 142B

What we are talking about here it clearly not the "Number One", but something like a set with no objects in it Z={ }.

- After this. The Philosopher Kant wrote several fascinating books on the relation of science and mathematics to philosophy. Up to this time He defined what he called analytic and synthetic logical propositions. Analytic propositions were defined. A synthetic proposition cannot be established without appeal to something other than the content of the concepts involved. He also distinguished between what he called "a priori" and " a posteriori" propositions. The question became "what kind of a proposition is "1 +1=2"? Is it a synthetic or analytic one? Some argue is it a third possibility, a "tautology". A tautology is a proposition true only formally because of variable substitutions. Kant did not believe like Plato and Frege was to later that numbers were "mental objects". He believed they were "transcendental objects". If you believe this we have a possibility that "1+1=2 is neither a synthetic proposition or a tautology, but a "logical truth." With the discovery of the non-Euclidean geometries in the second half of the nineteenth century this question became even more interesting. For we suddenly realized that there might be several different ways of giving "definition to the defined" as Pascal said. And, if this could be true in the fields of geometry, what about arithmetic? It wasn't until Frege brought back the idea from Plato that numbers are "objects" that it was solved in my opinion. As mentioned above, now there are actually three possibilities for this proposition, it can be a synthetically discovered truth from our intuition collectively agree upon (Henri Poincare later wrote a series of books proposing this view).

Hypothesis II) "If the One has being, it is one Entity with both unity and being.

This hypothesis turns out to have the consequences that assuming this:

142D "A One Entity is a whole of parts, (both one and many)."

This follows since 'is' is asserted to belong to the One, which is and

'one' is asserted to belong to this being which is One,

And since 'being' and 'one' are not the same thing.

This is what will be called "Unity in Diversity" later by our forefathers in the United States.

145A "A One Entity (having parts…its unity and being) is indefinitely numerous and also limited.'

This follows since 'unity' can never be lacking in its part 'being' nor vice versia.

Thus, each of the two parts never lacking in the other will co-define themselves forward inductively.

This argument will referred to as proof by co-induction two thousand years later [10] Doets and Van-Eijck

145A "A One Entity (being limited) can have both extension in space and shape."

Indeed, being limited it is a form (something defined or made up of objects satisfying certain

It could be a tautology (derived only from making formal substitions in the variables in the propositional formula). Or, it could be a logical truth which existed by itself for some reason more than a tautology. In order to have "logical truths" be something more than tautologies we have to start out with objects to subsitute in the formulas that are "real in some sense" and whose existence tells us something more than a tautology does. This is how the mathematical area of "set theory" came to be. Frege was the first to found it using both the logical formulas and syllogisms of Aristotle (see my talk last year on the history of "logic machines") and the philosophical arguments of Kant for its methodological thinking.

So, there is some confusion introduced here by Plato in that he proves his assertion assuming his "One Entity" is what we now call "The Number One" but the previous assertions were proved that it was just, the broader "Concept of One" or "Oneness".

149D "The One Entity, as described, has and has not contact with itself and Others."

This assertion again, is proved, by assuming we are talking about, "The Number One".

The word contact means the reality of the 'participation of forms' (mental objects) in each other as already postulated in other dialogues. Thus we can say that two ideal lines can intersect or cross in space as mental objects, even though they are both idealizations of material things.

It is here that the writer of the dialogue attempts to explain how rational numbers are generated from the starting point of the Monad and Dyad explained above. However, in my opinion, it wasn't until much later in the late 19th century AD that Dedekind cuts were used to define real numbers as the 'gaps' between an upper and lower series of approximating rational fractions that this difficult passage was really understood by anyone. E. Landau,

- It could be a tautology (derived only from making formal substitions in the variables in the propositional formula). Or, it could be a logical truth which existed by itself for some reason more than a tautology. In order to have "logical truths" be something more than tautologies we have to start out with objects to subsitute in the formulas that are "real in some sense" and whose existence tells us something more than a tautology does. This is how the mathematical area of "set theory" came to be. Frege was the first to found it using both the logical formulas and syllogisms of Aristotle (see my talk last year on the history of "logic machines") and the philosophical arguments of Kant for its methodological thinking.
- Frege in his book "Foundations of Arithmetic" ask the question, "How do we define the number one."? Up to this time nobody thought this was necessary. But, if we assume it is an object of thought (not just a concept or a meaning of concept) then it must be possible to have it be an element in a set. What can this element be? For Frege this element was the most fundamental element one could have in a set as an object, other than the element "zero". He had taken the object "zero" to be represented by the set which had no elements. So the next most fundamental element must be this set itself which has as its only element an "empty set".

The proof of this was delayed until the 17th and 18th centuries when mathematicians created the foundations of the theory of infinite series.

Define . Using the mathematical procedure called proof by induction, we may demonstrate that for all positive integers n. Indeed, this is true for the case n=1 because T = T. And, if we know it to be true for the case n=n-1, we know that . Then, adding the value of to both sides of this equation gives us the assertion for the case n=n.

Since, as n increases indefinitely, this shows that .

The fact that this proof uses the "principle of induction" which itself assumes there is a first element in the series of times brings us back to the importance of the "number one" again.

Zeno's paradox arises from not understanding the statement above from section 151B in the dialogue "The One Entity (as continuous quantity and magnitude) is equal and unequal both to itself and to the Others."

Earlier Zeno and Gorgias had said: "an unlimited being (a many) cannot either be in itself or in something other than itself, namely place. And, concluded that it must be nowhere." "and, if it is nowhere it does not exist and place does not exist and if things are many, there do not exist."

This was because it seemed that there were two equally possible but logically

- It wasn't until later in the 20th century that Dr. John Von Neumann proposed a different way to define the number one, here is a good short explanation from a recent discussion I had on the History and Philosophy of Science division on the Linkedin website of the difference between these two defintions:
- 1) Anatoly Tchoussov • to Harrell: may be I've said not very clear;
  I've meant that:
  Yours expression is not correct, because numbers are defined as classes of equivalencies and not as a sum of sets;

  2) such definition has an intrinsic difficulties, i.e. two non-equal models:
  in former note I will use Z as a symbol of an empty set, and figure brackets {} as symbols of a set;

  there are (at least) two ways to introduce numbers:

  A): {Z}, {{Z}}, {{{Z}}}, {{{{Z}}}}, ...

  B): {Z}; {Z, {Z}}; {Z, {Z}, {Z, {Z}}}; {Z, {Z}, {Z, {Z}}, {Z, {Z}, {Z, {Z}}}}; ...

  in a case A "3" doesn't belong to "5", but in a case B "3" belongs to "5";

- The beating heart of modern day computers is one and zero. As mentioned above Leibnitz and Boole made important conceptual discoveries that helped us do this. Colossus, one of the first computers developed in Britian, the mathematics of one and zero may have helped shorten World War II by as much as two years. So, to conclude, today, Roman numerals have been consigned to the dust bin of history. Pythagoras' idea of one and zero are all we need to create the modern computations that have transformed our world into a new information age and pushed humankind forward up to the edge of discovering how we ourselves are made (by God) genetically out of a coded string of amino acids.

- THE QUESTION'S RELATION TO THE MATHEMATICS OF SET THEORY

- The mathematical area of research called set theory was created in order to understand what "a real number" in Calculus means. This interest developed because in the previous decades techniques were developed in order to solve practical problems in mathematical analysis which made use of what Cauchy and Gauss (and earlier Euler) had called "complex" or "imaginary" numbers. Euler used an algebra of calculation in his trigonometric formulas (which had applications of mapping and geodesy) which made use of the imaginary number "i". Cauchy further developed these algebraic techniques and also showed how it was possible to integrate functions involving complex numbers. Gauss developed the beginnings of the concept of a "manifold" which would later revolutionize thinking in electrodynamics and Einsteinian physics. So the questions then became, "What is this 'real number' which determines how we calibrate or measure the space we are analyzing?" "What is a real function?" "What is a complex function?"

- Of course, for centuries and millennium philosophers had speculated about various theories of reality and metaphysics. But, in order to answer this question in an scientific and analytic sense people began thinking about what particular logical mathematical foundations that we had up to now assumed as given in the background of our axiomatic system determine its solution. Various theories of generalized algebraic numbers were created and Leopold Kronecker made his famous statement, "God created the integers and all else is the work of man.". But, how did God create the integers? People noticed that the positive integers formed what was called in set theory a "sequence" and that one of the main ways things were proved in mathematic problems involving sequences of integers was something called the "inductive principle". If a statement or proposition about integers was true for the next integer, after a given integer (no matter what that integer was) and it was true for the first integer, then it had to be true for all integers. The self-evident truth of this fundamental principle of mathematical proof of course depends on the fact (which is not true for all sets of objects) that there is a least positive integer, "One". So, how do we define the "number One" using modern propositional logic?
- Here are some more selections from the recent dialogue I had on the Linkedin website where some of the complexities of this question came out.

- Steve Faulkner • The number one is the mathematical object that leaves any number unchanged, under multiplication.
- Andrew Harrell • Steve,
  Yes, it is that, But, does that property define it uniquely? There is an object that belongs to the set of rationals and has this property. There is an object that belongs to the set of real numbers and has this property. There is an object that belongs to the set of integers and has this property. There are also three more objects that have the property of leaving any number unchanged under addition. Each of these objects have to be defined differently because because those different sets and different operations are defined differently. The question is how do we define one object that does all of this and is unique?
- Andrew Harrell • @Steve,
  A question you did not ask, but is pertinent is, 'If we define the Number One as the "set of all sets which have the set of no element in them", then how can it be a mathematical object, an "operator" which leaves any number invariant when multiplied by it? The answer I believe is because we have things called "Functors" from the category of sets to the category of arithmetrical operators. Functors were introduced in mathematics alot in the 1950s and 1960s in algebraic topology and algebraic geometry to computer mathematical characteristics of manifolds. However, I don't believe they were used in computer science much until recently with the Haskell computer language which has things called "Monads". ?

- A SHORT MATHEMATICAL ANSWER TO THE QUESTION
- **Here is the most generally accepted mathematic answer, figured out by the mathematician/philosophers Frege, Betrand Russell, and Peano at the turn of the beginning of the last century. In short… The number One is "the cardinality (similarity class of one-one functions) of the set whose only member is the empty set." This definition hides a huge logical complexity of definition. What is a similarity class? What is a one-one function (what is a mapping or function for that matter?). What is a set? What is an empty set (basically this is determined logically when you know what an element in a set is and what a set is)?**
- So what are the philosophical implications of this definitions? First of all in order to understand what the concept of Oneness is we have to understand what logic is. There must be an intellectual component (ie not only intuition to our theory of knowledge). And, we must understand what reality and the reality of an object is or means (for it to be an element in a set) in terms of the aforementioned intellectual component of our theory of knowledge. Then, if we understand this we must still also understand what functional computation (which allows us to create one-one mappings) is. This, in turn, allows us to understand what mathematical and scientific/theological Oneness is. We must understand this in order to understand what these two things are and the important fundamental question to our ethical and philosophical theories of well-being and truth.

- FURTHER POINTS ABOUT ITS THEOLOGICAL AND METAPHYSICAL IMPORTANCE

- But, what about its theological importance? Is it in fact the case as Dr. Kronecker has said, "God created the integers (and the science of nature that depends on using them to count and measure) and all else is the work of man." If an understanding of Oneness in terms of natural science is all that we want, then the definition of One from the above paragraph in set theoretical logical terms: "The number One is the cardinality (similarity class of one-one functions) of the set whose only member is the empty set." Solves the problem. With an understanding of this definition we have understood how God created the idea of "Oneness" inside of his created Universe. But, the Bible says that God did more than just create the World. It says that he created Man (and also Woman) in His own image and likeness. How does this relate to the above proposed set theoretical/logical definition of what "the number One" is? In the time between the beginning of the last century and our new millenium Mathematicians and logicians have shown, except for some notable gaps, how "real numbers (rational, algebraic, transcendental)", and likewise various other "complex and ideal numbers" can all be constructed logically from the positive integers. The possibility of the "notable gaps" come from the proof of the independence of the continuum hypothesis.

Concept as a Logical Relationship

- Here is another definition of a Concept to help us further elaborate this:
- ----- A concept is a logical relationship involving a predicative statement (subset of n times Cartesian product of the domain values and variables, instead of just a functional mapping). This logical relationship may also involve the question of the satisfaction of the concept (truth in terms of a specific knowledge representation). It may also involve the notion of a set of variable identifications in some model [data + algorithm][1]. And, it may also involve the notion of how a method for determining truth searches through the space of variable identifications inside of a pre-determined set of program search rules [logic + control][2] as a part of determining what the algorithm used will be.

This is another of those difficult philosophical problems mentioned at the beginning of this research paper. How can we understand what the "Concept of a Concept" is if we don't understand what the Concept of One is? This basic question is phrased above as a mathematical question in the logical foundations of the branch of mathematics called set theory. We believe the question of how we can develop a better understanding of science and faith is interconnected to this question. Thus, it is not only its mathematical implications that make this conundrum important for us to solve. This question from mathematical research relates to a fundamental theological and spiritual one… If we are to say that the fundamental nature of God is that He is One, what do we mean by this? The standard theological answer to this question is contained in a religion's teaching about the name of God (which is a trinity for Christians and Hindus). Is it possible to be a Father and Mother of God using structures of thinking within His Holy Spirit operating in our own minds and spirits and souls? And, an understanding of how our human interest in this question as a mathematical one developed will certainly better help us understand how our different human forms of science and faith are related in us.

To do this we need to go back to the turn of the last century when questions of the foundations of the area of mathematical analysis and the beginnings of the development of mathematical logic as it relates to the invention of computers was being developed. The mathematical area of research called set theory was created in order to understand what "a real number" in Calculus means. This interest developed because in the previous decades techniques were developed in order to solve practical problems in mathematical analysis which made use of what Cauchy and Gauss (and earlier Euler) had called "complex" or "imaginary" numbers. Euler used an algebra of calculation in his trigonometric formulas (which had applications of mapping and geodesy) which made use of the imaginary number "i". Cauchy further developed these algebraic techniques and also showed how it was possible to integrate functions involving complex numbers. Gauss developed the beginnings of the concept of a "manifold" which would later revolutionize thinking in electrodynamics and Einsteinian physics. So the questions then became, "What is this 'real number' which determines how we calibrate or measure the space we are analyzing?" "What is a real function?" "What is a complex function?"

**Of course, for centuries and millennium philosophers had speculated about various theories of reality and metaphysics. But, in order to answer this question in an scientific and analytic sense people began thinking about what particular logical mathematical foundations that we had up to now assumed as given in the background of our axiomatic system determine its solution. Various theories of generalized algebraic numbers were created and Leopold Kronecker made his famous statement, "God created the integers and all else is the work of man.". But, how did God create the integers? People noticed that the positive integers formed what was called in set theory a "sequence" and that one of the main ways things were proved in mathematic problems involving sequences of integers was something called the "inductive principle". If a statement or proposition about integers was true for the next integer, after a given integer (no matter what that integer was) and it was true for the first integer, then it had to be true for all integers. The self-evident truth of this fundamental principle of mathematical proof of
course depends on the fact (which is not true for all sets of objects) that there is a least positive integer, "One". So, how do we define the "number One"?**

The Greek philosophers realized this as a fundamental philosophical question too. Plato's dialogue Parmenides is perhaps his hardest and most important attempt in the classical era to try and understand this . And, it deals with just this question. "What is does the Concept of One mean

2^{nd} Order and 1^{st} Order Logical Definitions

- 1^{st} order logical definitions of sets of objects: They are of the form a set of objects = { x | x exists and satisfies a propositional logic functional predicate f(x)}
- 2^{nd} order logical definitions of sets of objects: They are of defining first the form a relational function which to be applied to a of objects =
- Set of pairs of Function elements -> (x,y) such that x exists and f(x)=y.
- The pairs determine the relationship we are trying to characterize and also a function f(x) is determined from the pairs if the mapping is injective or one-to-one and surjective or onto the whole range of the 2^{nd} part of the pair space. See the reference 10] for much more on this.
- 'If we define the Number One as the "set of all sets which have the set of no element in them", then this is a 2^{nd} order predicative definition. A problem arises in this, as explained by Hilbert [7]. The way sets defined by 2^{nd} order predicates are determined to be equal is by checking the functional values on all the terms of set elements in the basic universe of objects.

VII

CONCEPTS AS RELATIONS.
HOW CONCEPTS CHANGE,
WHILE REMAINING THE SAME,
INSIDE OF THE STREAM OF CONCIOUSNESS
IN OUR MINDS

This way of looking at concepts assumes that objects are appearing to us and the intellectual faculties in our minds are able to identify and unify how the objects fit together as time progresses.

METHODS AS CONCEPTS ----

While rules are used for backward directed goal-oriented reasoning, objects and recursively defined data types are appropriate for building up forward directed production systems, models are appropriate for procedural oriented, cased-based reasoning.

A model deals with some topic, a pattern of behavior, a procedure for accomplishing a taks, an overall type of reality (World view).

A paradigm or case is:

1) a way of looking at a body of facts
2) an example, a particularly good example
3) a pattern, an all encompassing pattern.

One can mistake a paradigm for a theory - in the same way one can mistake a series of examples for a definition. A good example (a paradigm) can serve as a model for the interpretation of a body of facts. However, when it becomes a model it becomes capable of being displace. Remaining a

- Theoretically, there are an infinite number of possible objects, But, when we start out the calculation to check for equality we only have a finite set of data values. When Hilbert and Ackermann wrote their book on mathematical objects the techniques computer programming was just being thought out. They thought that since these considerations keep up from using formal logic to prove the existence of an infinity of numbers we couldn't compute things determined this way. It was not conceived (until Curry and Church thought it out later) that we could have dynamic memory allocation of things like what are called now "streams" in functional programming. Functional programming data streams applied to sets assume what we call in set theory as "The Axiom of Infinity" as justification for reasoning inductively and forward-chaining logically ahead for purposes of computation. The reason this works is basically the same reason that "Zeno's Paradox" does not keep us from defining the real continuum.

--- A model can be a way of looking at the World [a physical theory, ethical theory, philosophic theory, religious theory]. In this case there is a interpretation [mapping] from the objects in the World to a set of facts, constants, variables. There is a translation of physical laws, ethical, philosophic, religious beliefs and postulates into rules connecting those facts, constants and variables.

IMPORTANT NOTE: The more we want to talk about knowledge related to the World and the less about knowledge related to ethics, philosophy, theology, the more we need to introduce numbers and their language mathematics into the propositions we write. For example, once we have numeric data types, we can go from simple verbal propositions (qualitative judgements) to statements involving numeric values (quantitative judgements). We can then make a philosophic classification type expert system of the type given in the discussion of the second definition of a concept, into a quantitative case-based reasoning tool.

--- In a classification type expert system, the model involves a data structure as in the 3rd definition of a concept. It also involves some means of retrieving the information, along with a way of creating the rules as in the 2nd definition of a concept.

How do we create the knowledge tree that we use in such a quantitative case-based reasoning expert system? For the philosophic expert system we asked a series of questions from the general to the specific about something we believe that we already have in the mind. Now, for this, we need a series of examples or cases in order to develop cutoff values of object attributes in order to branch into the knowledge tree. Again the questions go from the general to the specific. But how do we know which attributes are general. Answer: we can construct statistical summaries and tables that analyze the examples in the data to determine which attributes are most associated which the particular results we are interested in

Frege's and Von Neuman's 1st Order Definitions

- Let the Number Zero be the set whose only element is the empty set:

Z= Zero = { x | x does not exist in any set}

One = { Z}, the Two is defined as {One} or {{Z}}

Alternatively, according to Von Neumann, for ordinals we can say Two = {Z, {Z}}

As mentioned above, Kant believed that numbers were "transcendental objects" not "mental objects" as Plato did. This definition combines and unifies the two ways of looking at numbers as well as introducing the machinery of Aristotelian logic and syllogisms further into Kant's ideas. Investigating the subtle but fascinating distinctions between these two equivalent ways of defining one (as an cardinal and ordinal) was to occupy the though of Turing at the same time he helped invent theoretically our modern day digital computer.

NOTE: This learning procedure does not require that the sample space be partitioned into (object-oriented) categories first. Nor, does it require that there be a top-down identification tree of the twenty question type. However, this procedure can be used in combination of one or another of the other procedures.

As will be explained in the next section the order of the steps in this algorithmic approach may be used not only for neural network type classification programs, but also for logic programming expert system classification systems. These programs ask a series of questions to an outside person running the program (human expert) and based on his or her answers, along with the set of rules built into the system output a set of conclusions as each rule in the ruleset fires. According to a theorem proved by Jacques Herbrand[27] the logical system represented by the statements the rules make can, under most circumstances (where there is a finite set of rules and the rule base logic is monotonic) be assumed to be certain type of simplified statements called clauses.

[1] Martin Gardner, Logic Machines and Diagrams, 1st edition, 1958, page 3.

[2] Gardner, op cit. page 10.

[3] Gardner, op cit, page 12.

[4] This, of course, assumes an Aristotelian view of metaphysics in which "all things are good and true." Goodness, being in scholastic philosophy, that which contains within itself its own purpose,

[5] Gardner op cit. pg 31.

[6] Symbolic Logic, John Venn, Cambridge U. Press 1881.

[7] Symbolic Logic, John Venn, 2nd edition, 1894 may be consulted for a clear exposition of Euler's system.

[8] Prior Analytics, Book 1, section 1.

Frege's Definition of Numbers using 2nd Order Logical Predicates

- Frege believed that numbers were objects. He also believed that Plato that concepts were objects. However, he did not believe that numbers were concepts. He believe that numbers were values or extensions of concepts. He wrote his Begriffsschrift (concept notation) lectures in 1879 in which we laid out the logical foundations for his idea of a Humean and Kantian theory of Leibnitzian identity in logical propositions . He also explained in this paper how these ideas could be used to give a better idea of what a the concept of "mathematical function" is. And, later he wrote "Basic Laws of Arithmetic" [17] in 1893 in which he attempted to formalize his above idea of numbers being "extensions of concept". Then, however, Bertrand Russell after reading the Basic Laws came up his paradox related to how Frege defined his values or extensions of concepts. To this day many logical positivist philosophers believe this was a "knockout blow" to Frege's ideas. However, in his lectures from 1910 [16] Frege leaves out his Axiomx V and VI from the Grundgesetze which led to the Russell paradox problem from his theory of extensions. What is left is a clear and workable system of mathematical logic in which set theory, a theory of identity in statements of propositional logic, mathematical functions, ordinal numbers, cardinal numbers can be defined.

[16] See Rudolf Carnap's notes on "Frege's Lectures of Logic" from 1910 at Jena (Published by Open Court).

As to the detals of how this relates to recursive functions, lamba calculus, functional computing; that is a lot harder to understand. Maybe we (the members of this discussion group) will be able to figure it out together?

Hilbert's 2nd Order Definitions of Zero and Equality with Class Predicates

- The mathematician David Hilbert in his book [7] makes the definition Zero = 0(F): ~(E x)F(x) as an "operator" and not a "set" which is a differrent defintion than Frege's

(which means verbally, there is no x for which F is true)

We can create a correspondence between "operators" F and "sets" S by associating the set of all elements s in S which make operators f in F true. Then, if we translate this definition into terms of sets, what it says is that Zero (as a set) = the set with no elements in it, what we have called Z above (so it is the empty set and this definition given here is same way Frege defined Zero). The reason is that if any element was in Zero, then applying the identity function to it under the above correspondence would contradict this.

- Definition: ==(x,y) : (F)(F(x)=>F(y)&&F(y)=>F(x))
- That is, he calls x identical with y, if any predicate which holds for x also holds for y and vice versia. [10]

By the time Gottfried Frege wrote his book "The Foundations of Arithmetic" logicians had still not figured out how to define the number one in terms of propositional logic. In fact on page 44 of his book after having recounted previous attempts, he notes that they have all failed.[1] He then considers the question of whether all units ('ones') are identical with another? "We cannot succeed in making different things identical by dint of operations with concepts. But, even if we did, we should no longer have things in the plural but only one thing; for as Descartes says, ' the number, or better the plurality in things arises from their diversity.' ..Jevons has said,'Number is but another name for diversity. Exact identity is unity, and with difference arises plurality.'...Leibnitz long ago rebutted the view of the schoolmen that number results from the mere division of the continuum'" Frege then goes on in his book to take the concept of zero and one as undefined in order to define the concepts of the rest of the natural numbers inductively from them.

He defines abstract number as, "the empty form of difference"[2]. "Number is not anything physical, but nor is it anything subjective (an idea)." "The content[meaning] of a statement of number is an assertion of a concept[3]."

The most generally accepted mathematic answer, figured out by the mathematician/philosophers Frege , Bertrand Russell, and Peano at the turn of the beginning of the last century can be explained fairly simply. The number One is "the cardinality (similarity class of one-one functions) of the set whose only member is the empty set." This definition hides a huge logical complexity of definition. What is a similarity class? What is a one-one function (what is a mapping or function for that matter?). What is a set? What is an empty set (basically this is determined logically when you know what an element in a set is and what a set is)? A more detailed explanation in symbolic terms will be stated shortly.

So what are the philosophical implications of this definitions? First of all in order to understand what the concept of Oneness is we have to understand what logic is. There must be an intellectual component (ie not only intuition to our theory of knowledge). And, we must understand what reality and the reality of an object is or means (for it to be an element in a set) in terms of the aforementioned intellectual component of our theory of knowledge. Then, if we understand this we must still also understand what functional computation (which allows us to create one-one mappings) is.

As mentioned above, the definition of what one-to-one function is in terms of 1st and 2nd order propositional logic was discovered by Frege. However, since his notation and formulas are rather hard to explain we will look at a shorter way of defining the same concept (that of the number One) due to Hilbert.[4] We make us of the definition of a concept given at the start of the section as a 2nd order logical predicates (relations whose arguments themselves are allowed to be relations or functions). Because we are working now with logical statements and not the Boolean functions discussed above we can write propositions such as:

Definition 0(F): ~(E x)F(x)

(which means verbally, there is no x for which F is true)

Or, as Frege would say using his definition of a number in terms of its content as an assertion about a concept: "The number zero belongs to a concept, if the proposition that a does not fall under that concept is true universally, whatever a may be."[5]

Definition: ==(x,y) : (F)(F(x)~F(y))[6]

Which is the way Hilbert defines the number zero and the relationship of predicate equality inside of 2nd order logic.[7] In words, he calls x identical with y, if any predicate which holds for x also holds for y and vice versa.

Hilbert's 2nd Order Definition of One as an Operator (a property of sets)

- The next definition of a number in Hilbert's book is a little harder to understand. Definition The Number One = 1(F): (Ex)[F(x) & (y) (F(y) implies ==(x,y)] as an "operator" [7].
- Verbally, this says, "There is an x for which F(x) holds, and any y for which F(y) holds is identical with this x." If we apply the correspondence between "operators" F and "sets" S of the previous slide we can see what set this corresponds to. It corresponds to a set S of elements x (which we are calling a set of elements "One") such that if the term exists and it is true that F(x) is true (can be verified for the propositional function F (x)) and it is also true that there is a term y such that F(y) is true, then x==y. Or, in other words, we define set as being determined by its elements. Then, we say, the "number one" is the set determined by only one element.
- So, this is a new way to do the definition is that is different than defining "One" = {{Z}} or "One" = {Z {Z}} and it uses functions as well as sets or "objects".
- Thus, this definition utilizes the idea of defining the number one as a Functor (see previous slide for definition of this). It defines the number as a "property of sets" instead of a "property of predicates". In Frege's book this is stated as, "The number one belongs to a concept F, if the proposition that a does not fall under F is not true universally, whatever a may be, and if from the propositions "a falls under F" and "b falls under F" if follows universally that a and b are the same."[1]

Once we have defined zero and equality we proceed with a definition of the next number "One".

Definition 1(F): (Ex)[F(x) & (y) (F(y) implies ==(x,y)][8].

Verbally, this says, "There is an x for which F(x) holds, and any y for which F(y) holds is identical with this x."

In Frege's book this is stated as, "The number one belongs to a concept F, if the proposition that a does not fall under F is not true universally, whatever a may be, and if from the propositions "a falls under F" and "b falls under F" if follows universally that a and b are the same."[9]

there exists a one-to-one correspondence between the elements that make up the two sets.

Hilbert's Definition of Two:

Also, the next number two (and all the rest of the natural numbers) can be defined using the same iterative procedure.

Definition The Number Two =

2(F): (Ex)(Ey) (~ (x,y) & F(x) & F(y) & (z)

[F(z) implies ==(x,z) or == (y,z)]

2^{nd} Order Definition of "One" as a set (a property of predicates)

- As we said above the definition of "One" as the "set of all sets which are equivalent to the set with only the empty set as a member" is a 2^{nd} order definition. Several things must be in our set theory in order to allow this definition. We must have an "axiom of infinity" , an "axiom of the universal set" V, and an ability to evaluate logical predicates functionally and allocate new terms dynamically. Here **we need to apply the correspondence between "equivalence classes of sets", ~(S) and "sets" S. The way 2^{nd} order predicates come in here is in the "equivalence class relations" that are defined by the operator definition in the previous slide. In order to specify "sets which are equivalent to the set with only the empty set as a member" we must have a universal set V in which these sets occur and are defined by logical predicates with respect to. In addition, we must be able to pick out elements from all sets and map equivalences between two different occurances of the "set with only the empty set as a member" in the universe V of all sets. This requires what is called the "Axiom of Singletons" to be true. It says that we assume that for every object x, the set {x} = {y such that y==x} exists. Here the equality of sets is defined element-wise (x==y if for all z existing in x, z exists in y and vice-versia . It is not defined as above where x == y, if and only if any predicate which holds for x also holds for y and vice versia. Given any "well-ordered" set it has a least element and this allows us check whether it is the only element in "one", the empty set in this model.**

"There are two different x and y for which F is true, and any z for which F(z) holds is identical either with x or with y. [7]

Also, the next number two (and all the rest of the natural numbers) can be defined using the same iterative procedure.

Definition 2(F): (Ex)(Ey) (~ (x,y) & F(x) & F(y) & (z)
[F(z) implies ==(x,z) or == (y,z)]

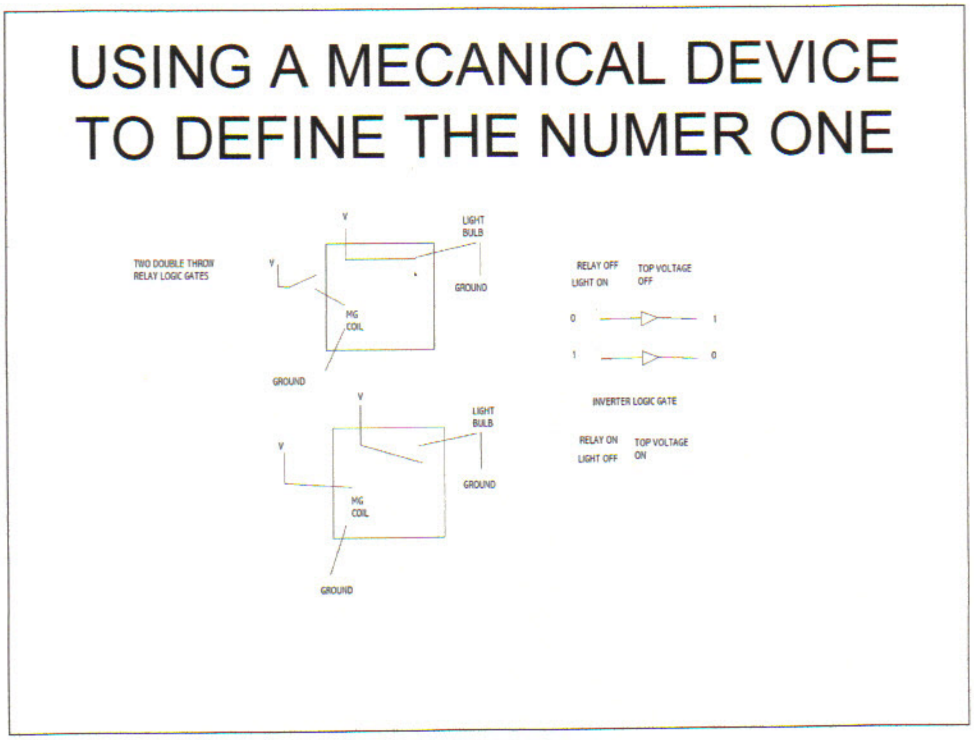
"There are two different x and y for which F is true, and any z for which F(z) holds is identical either with x or with y.

Examining the above definition we see that "If we are to use the symbol a to refer to or signify an object, we must a criterion for deciding in all cases whether b is the same as a, even if it is not in our power to apply this criterion."[10] An understanding of how to state the solution to this problem goes back to the English philosopher Hume: "When two numbers are so combined as that the one always has a unit answering to every unit of the other, we pronounce them equal."[11] When the idea of set, a correspondence (function) have been fully defined logically this can be stated more simply as saying

Quine's New Foundations for Set Theory

- **There is at least a third possibility, other than Frege's, Von Neumann's and Hilbert's, for defining the "Number One" In the 1940s and 1950s, the student of Bertrand Russell, Quine was a professor of the philosophy of mathematics and logic at Harvard. He published as theory of sets in which we can assume that the "set of all sets" necessarily exist as an axiom at the start. This universal set V is/was the same set as Plato and Parmenides discussed in their dialogue and called the set of "Others". But, he does not define "One" the same way Plato and Parmenides did. He defines it as meaning V, what Plato and Parmenides called "the Others". In this theory using several other more standard set theory axioms from the Zermelo-Frankel set theory it is possible to prove what we call the "Axiom of Choice" as a theorem of the theory and not an axiom. If you add another axiom, "The axiom of Singletons" this can happen. It says that we assume that for every object x, the set {x} = {y such that y==x} exists. The number Z "Zero" (or empty set) is defined as before. But, One, has been defined here as the set whose only element is V (the whole Universe of sets). The rest of the natural numbers can also be defined in this form of set theory, but, as you would think their definitions will have very different meanings.**

You might want to study Quine's book [13] or the notes of Randall Holmes[14] along with our discussions at the LinkedIn website referred to above for more on this and the relation of the different approaches to each other.



USING A MECHANICAL DEVICE TO DEFINE THE CONCEPT OF THE NUMBER ONE

In the above we defined the concept of the number One using words and other concepts. But, is it possible to take the viewpoint that everything is reducible to a materialistic interpretation while doing this? In the nineteenth century telegraph switches were experimented with in order to route signals, but there connection with the Boolean functions we have been talking about wasn't realized until half way into the twentieth century. As shown in the diagrams below it is possible to construct, fairly simply, using electromechanical relays devices which we later put together using vacumn tubes and transistors to become what we now call NAND logic gates:

In the figure above the current flowing through the electrical relays energies the switches that control whether the light bulb is shining or not (V represents voltage source, MG COIL represents the magnetic coil that moves the relay when it is activated). If we vary the input voltages in order to get different output values then the systems represents what we now call an INVERTER LOGIC GATE.

If we represent the high voltage condition of the input voltage sources of the two relays above as 1 and the low voltage condition as 0, and we vary the input values in order to get a Boolean valued output, then the total system represents what we now call a NAND logic gate.